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GENERALIZATION OF A SIMPLE FORMULA.

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§ 1. The inequality

$$(1)\qquad a^2+b^2 \geqq 2ab$$

is well known. The object of this paper is to give a simple proof of a generalization of this formula, viz., of the formula:

$$(\mathrm{I})\qquad C_{n,j} \sum_{i=1}^{n} a_i^j \geqq n \sum_{k_1,\, k_2,\, \dots,\, k_j=1}^{n} a_{k_1}\, a_{k_2} \dots a_{k_j},$$

for $j \leqq n$, where all quantities a are real and positive, j and n are positive integers, and

$$C_{n,j} \equiv \frac{n!}{j!\,(n-j)!}.$$

We shall furthermore always assume:

$$k_1 < k_2 < \dots < k_j.$$

As special cases, we have:

$$j=2,\ n=2,\ a_1^2+a_2^2 \geqq 2a_1\, a_2$$

$$j=2,\ n=3,\ a_1^2+a_2^2+a_3^2 \geqq a_1\, a_2+a_2\, a_3+a_3\, a_1$$

$$j=2,\ n=4,\ 6(a_1^2+a_2^2+a_3^2+a_4^2) \geqq 4(a_1\, a_2+a_1\, a_3+a_1\, a_4+a_2\, a_3 + a_2\, a_4+a_3\, a_4)$$

$$j=3,\ n=3,\ a_1^3+a_2^3+a_3^3 \geqq 3a_1\, a_2\, a_3.$$

§ 2. We shall first prove that (I) holds for $j=p+1$ and $n=p+1$, if it holds for $j=n=p$; *i. e.*, if we think of the positive integers arranged as double indices in a square matrix, we shall prove the formula to hold for the elements along the principal diagonal.

We have for $n=j=p$, since $C_{p,p}=1$:

$$\left\{\begin{array}{l}
a_1^p+a_2^p+\dots+a_p^p \geqq pa_1 a_2 \dots a_p \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
a_1^p+\dots+a_{i-1}^p+a_{i+1}^p+\dots+a_{p+1}^p \geqq pa_1 \dots a_{i-1}\, a_{i+1} \dots a_p \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
a_2^p+a_3^p+\dots+a_p^p+a_{p+1}^p \geqq pa_2 a_3 \dots a_p\, a_{p+1}
\end{array}\right.$$

Clearly, there are $p+1$ such inequalities, one element out of the $p+1$ being omitted in each of them. Hence, if we add them, each term a_i^p ($i=1$

$\ldots p+1)$ will occur p times on the left hand side, whereas on the right-hand side, we shall have the sum of all possible products of p elements out of the $p+1$. We obtain then, after dividing by p:

$$\sum_{i=1}^{p+1} a_i^p \geqq \sum_{\substack{i_1, i_2 \ldots i_p = 1 \\ i_1 < i_2 \ldots < i_p}}^{p+1} a_{i_1} a_{i_2} \ldots a_{i_p}$$

Multiplying both sides of this inequality by $\sum\limits_{i=1}^{p+1} a_i$, we get, since all a_i's and therefore $\Sigma\, a_i$ are positive:

$$\sum_{i=1}^{p+1} a_i^{p+1} + \sum_{\substack{i, j=1 \\ i<j}}^{p+1} a_i\, a_j\, (a_i^{n-1} + a_j^{n-1}) \geqq (p+1) \prod_{i=1}^{p+1} a_i + \sum_{\substack{i, j=1 \\ i<j}}^{p+1} (a_i^2 + a_j^2) \prod_{k=1}^{p+1}{}'' a_k,$$

where $\prod\limits_{k=1}^{p+1}{}'' a_k \equiv a_1 a_2 \ldots a_{i-1} a_{i+1} \ldots a_{j-1} a_{j+1} \ldots a_{p+1}$, *i. e.* the product of all a_i's $(i=1\ldots p+1)$, except a_i and a_j. Now, we know from (1):

$$a_i^2 + a_j^2 \geqq 2a_i\, a_j.$$

Hence, we get:

$$\sum_{i=1}^{p+1} a_i^{p+1} + \sum_{\substack{i, j=1 \\ i<j}}^{p+1} a_i\, a_j (a_i^{n-1} + a_j^{n-1}) \geqq (p+1) \prod_{i=1}^{p+1} a_i + 2C_{p+1,\,2} \prod_{i=1}^{p+1} a_i.$$

Furthermore, we have

$$(a_i^p - a_j^p)(a_i - a_j) \geqq 0.$$

Hence, $a_i^{p+1} + a_j^{p+1} \geqq a_i\, a_j (a_i^{p-1} + a_j^{p-1})$. Therefore, we may write:

$$\sum_{i=1}^{p+1} a_i^{p+1} + \sum_{\substack{i, j=1 \\ i<j}}^{p+1} (a_i^{p+1} a_j^{p+1}) \geqq (p+1) \prod_{i=1}^{p+1} a_i + 2C_{p+1,\,2} \prod_{i=1}^{p+1} a_i.$$

On the left-hand side, every a_i^{p+1} $(i=1\ldots p+1)$ occurs now $p+1$ times; remembering, therefore, that

$$2C_{p+1,\,2} = (p+1)p,$$

we obtain, after dividing through by $p+1$:

$$\sum_{i=1}^{p+1} a_i^{p+1} \geqq (p+1) \prod_{i=1}^{p+1} a_i,$$

which is the desired result, since also $C_{p+1,\,p+1}=1$.

§ 3. Secondly, we shall prove that if (I) holds for $j=p$ and $n=m$, it will also hold for $j=p$ and $n=m+1$; *i. e.*, we shall prove the formula for the double indices above the principal diagonal of their matrix. We have now:

$$\left\{\begin{aligned}
&C_{m,p}(a_1{}^p+\quad\ldots\ldots\ldots\quad+a_m{}^p)\geqq m\sum_{k_1\ldots k_p=1}^{m}a_{k_1}\,a_{k_2}\ldots a_{k_p},\\
&C_{m,p}(a_1{}^p+\ldots+a_{i-1}{}^p+a_{i+1}{}^p+\ldots a_{m+1}{}^p)\geqq m\sideset{}{'}\sum_{k_1\ldots k_p=1}^{m+1}a_{k_1}\,a_{k_2}\ldots a_{k_p},\\
&C_{m,p}(a_2{}^p+\quad\ldots\ldots\ldots\quad+a_{m+1}{}^p)\geqq m\sum_{k_1\ldots k_p=2}^{m+1}a_{k_1}\,a_{k_2}\ldots a_{k_p},
\end{aligned}\right.$$

where $\sideset{}{'}\sum\limits_{k_1\ldots k_p=1}^{m-1}$ indicates that $k_1, \ldots, k_p$ are to take all values from 1 to $m+1$, *except i.*

We have this time $m+1$ inequalities, each one of the elements $a_1, \ldots, a_{m+1}$ being omitted in one of the inequalities. Hence, if we add them, each term a_i^p $(i=1, \ldots, m+1)$ will occur m times on the left-hand side. On the right-hand side of each inequality, there are $mC_{m,p}$ terms; in the sum there willbe $(m+1)mC_{m,p}$ terms, each of which is a product of p elements out of the $m+1$. Of those, there are only $C_{m+1,\,p}$ different ones. Since there is symmetry with respect to every element, each of these products is repeated $\dfrac{(m+1)mC_{m,p}}{C_{m+1,\,p}}$ times. Hence, we obtain:

$$mC_{m,p}\sum_{i=1}^{m+1}a_i{}^p\geqq\frac{(m+1)mC_{m,p}}{C_{m+1,\,p}}\prod_{i=1}^{m+1}a_i{}^p\ \text{ or }\ C_{m+1,\,p}\sum_{i=1}^{m+1}a_i{}^p\geqq(m+1)\prod_{i=1}^{m+1}a_i{}^p,$$

which was to be proved.

The results of § 2 and § 3 establish the formula, which it was desired to prove.

§ 4. In part II of Chrystal's *Algebra* (second edition, 1906), we find on page 46:

$$\frac{a_1+a_2+\ldots+a_k}{k}\geqq(a_1\,a_2\,\ldots\,a_k)^{1/k},$$

and on page 49:

$$\frac{a_1{}^m+a_2{}^m+\ldots+a_k{}^m}{k}\geqq\left(\frac{a_1+a_2+\ldots+a_k}{k}\right)^m.$$

Combining these two formulae for $k=m$, we obtain (I) for $j=n$. The general formula (for $n\geqq j$) is not in the literature so far as I know. It might still be desired to establish a formula which shall hold also for $n<j$.